Exam Computational Fluid Dynamics

January 29, 2015. Duration: 3 hours.

The weights used to determine the final mark are given below. The maximum score is 36 + 4 (free) = 40 points.

Problem 1

Let the difference operator C be defined by $C\phi_j = (\phi_{j+1} - \phi_{j-1})/(2h) - \lambda(\phi_{j+1} - 3\phi_j + 3\phi_{j-1} - \phi_{j-2})/h$, where h is the mesh size, $x_j = jh$ and $\phi_j = \phi(x_j)$.

a. [3 points] Show that the difference operator $C\phi_j$ is for general λ a second-order accurate approximation of $\phi'(x_j)$, and third-order accurate only if $\lambda = 1/6$.

b. [3] Give the general solution of the difference equation $C\phi_j = 0$. What happens with this general solution if λ tends to zero?

c. [2] We want to solve the problem $\phi'(x) = 0$ for $x \geq 0$ with $\phi(0) = 1$. How can this boundary condition be added to the difference scheme such that all freedoms of the general solution are fixed?

d. [2] Determine the symmetric and skew symmetric part of the above difference operator for general λ and determine from that the artificial diffusion of the scheme.

Problem 2

Consider the equation

$$\frac{d}{dt}\phi_j(t) + \frac{u}{2h}(\phi_{j+1}(t) - \phi_{j-1}(t)) - \frac{k}{h^2}(\phi_{j+1}(t) - 2\phi_j(t) + \phi_{j-1}(t)) = f_j(t),$$

for $j = 1, ..., M$ and where $h = 1/M$. Here $\phi_0(t) = \phi_M(t) = 0$ for all t and $\phi_j(0) \geq 0$ for all j and $f_j(t) \geq 0$ for all j and t.

a. [3] Apply the Forward Euler method to this problem and show that for a mesh-Péclet number less than 2 we get a positive solution for all time if the time step satisfies a certain criterion. Give this criterion.

b. [3] Compute the Fourier amplification factor of the scheme of part a and give the condition on the time step which makes the computation absolutely stable.

Exam questions continue on other side

Problem 3

a. [2] Show that the operator L implicitly defined by $L\phi = (u\phi)_x + (v\phi)_y$ is a skew adjoint operator if $\text{div}\,\mathbf{u} = 0$, i.e. show that $\int_\Omega \psi L\phi d\Omega = -\int_\Omega \phi L\psi d\Omega + \int_\Gamma ..$ for any domain Ω with boundary Γ.

b. [1] Suppose we have the equation $\phi_t = L\phi$ defined on the unit square with L as defined in the previous part and $\phi(x, y, 0)$ given. What is relevance of the skew-adjointness of L for the solution $\phi(x, y, t)$ of this equation?

c. [3] Consider an uniform mesh. Give a discretization of the operator L that preserves the skew-adjointness of L.

Problem 4

a. [3] In the Lecture Notes, three different positionings are discussed for the variables occurring in the incompressible Navier-Stokes equation. Sketch these three positionings. If we use central second-order differences for the gradient operator acting on the pressure p, what kind of indeterminacy occurs for the pressure in each of these cases?

b. [3] Consider the staggered arrangement of velocities and pressures as in the Marker-and-Cell (MAC) method. Derive the pressure Poisson equation that arises when the time-dependent incompressible Navier-Stokes equations are discretized using an explicit method.

c. [2] Show that the matrix occurring in the equation of part b is singular if on all boundaries the normal velocity is prescribed and give also the associated eigenvector.

d. [2] Suppose we want to simulate the flow around a wing-like shape in a wind tunnel. Assume the flow is incompressible. Indicate all the boundary conditions you would impose, and motivate them.

Problem 5

a. [2] The smallest space scale l_K and time scale τ_K in a turbulent flow can be expressed in the properties of the largest eddies in the flow by $l_K = \text{Re}_{ed}^{-3/4} l_{ed}$ and $\tau_K = \text{Re}_{ed}^{-1/2} l_{ed}/u_{ed}$, respectively. How do these relations lead to the statement that if for turbulent flow the Reynolds number increases by a factor 10 that the complexity of the computations is increased by about a factor 1000.

b. [2] For the comparison of results of two different numerical models for the same turbulent flow problem, one cannot just compare the two solutions at a certain time instance, why not? Which two quantities are compared?